Exercise 39

Find the derivative. Simplify where possible.

$$g(t) = t \coth \sqrt{t^2 + 1}$$

Solution

Take the derivative using the chain and product rules.

$$\begin{aligned}
g'(t) &= \frac{d}{dt}\left(t \coth \sqrt{t^2+1}\right) \\
&= \left[\frac{d}{dt}(t)\right] \coth \sqrt{t^2+1} + t\left[\frac{d}{dt}\left(\coth \sqrt{t^2+1}\right)\right] \\
&= (1) \coth \sqrt{t^2+1} + t\left[\left(-\operatorname{csch}^2 \sqrt{t^2+1}\right) \cdot \frac{d}{dt}\sqrt{t^2+1}\right] \\
&= \coth \sqrt{t^2+1} + t\left[\left(-\operatorname{csch}^2 \sqrt{t^2+1}\right) \cdot \frac{1}{2}(t^2+1)^{-1/2} \cdot \frac{d}{dt}(t^2+1)\right] \\
&= \coth \sqrt{t^2+1} + t\left[\left(-\operatorname{csch}^2 \sqrt{t^2+1}\right) \cdot \frac{1}{2}(t^2+1)^{-1/2} \cdot (2t)\right] \\
&= \coth \sqrt{t^2+1} + t\left[\left(-\operatorname{csch}^2 \sqrt{t^2+1}\right) \cdot \frac{t}{\sqrt{t^2+1}}\right] \\
&= \coth \sqrt{t^2+1} + t\left(-\frac{t \operatorname{csch}^2 \sqrt{t^2+1}}{\sqrt{t^2+1}}\right) \\
&= \coth \sqrt{t^2+1} - \frac{t^2 \operatorname{csch}^2 \sqrt{t^2+1}}{\sqrt{t^2+1}}
\end{aligned}$$